[This question paper contains 6 printed pages.]

(11) Your Roll No. 2022

Sr. No. of Question Paper : 738 **B**

Unique Paper Code : 32351201

Name of the Paper : BMATH203 – Real Analysis

Name of the Course : **B.Sc. (H) Mathematics**

Semester : II

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. All Questions are Compulsory.
3. Attempt any **two** parts from each question.
4. All Questions are of equal marks.

1. (a) State the completeness property of $\mathbb{R}$, hence show that every non-empty set of real numbers which is bounded below, has an infimum in $\mathbb{R}$.

P.T.O.

(b) Show that if A and B are bounded subsets of $\mathbb{R}$, then $A \cup B$ is a bounded set and $\sup (A \cup B) = \max \{\sup A, \sup B)$.

(c) State and prove nested interval property.

(d) Define an open set and closed set in $\mathbb{R}$.

Show that if $a, b \in \mathbb{R}$, then the open interval (a, b) is an open set.

Is a closed interval a closed set ?

2. (a) Let S be a bounded set in $\mathbb{R}$ and let S_0 be a non-empty subset of S. Show that

$$\inf S \leq \inf S_0 \leq \sup S_0 \leq \sup S$$

(b) State Archimedean property. Hence, prove that if

$$S = \left\{\frac{1}{n}, n \in \mathbb{N}\right\} \text{ then } \inf S = 0.$$

(c) If $S \subseteq \mathbb{R}$ is non empty. Show that S is bounded if and only if there exists a Closed bounded interval I such that $S \subseteq I$.

(d) If x, y, z $\in \mathbb{R}$ and $x \leq z$. Show that $x \leq y \leq z$ if and only if $|x - y| + |y - z| = |x - z|$. Interpret this geometrically.

3. (a) Prove that a convergent sequence of real numbers is bounded.

Is the converse true? Justify.

(b) Let (x_n) be a sequence of positive real numbers such that $\lim_{n\to\infty}\left(\frac{x_{n+1}}{x_n}\right) = L$ exists. If $L < 1$, then (x_n) converges and $\lim_{n\to\infty}(x_n) = 0$.

(c) Prove that if $C > 0$, then $\lim_{n\to\infty}\left(C^{1/n}\right) = 1$.

(d) Let $x_1 > 1$ and $x_{n+1} = 2 - \frac{1}{x_n}$ for $n \in \mathbb{N}$. Show that (x_n) is bounded and monotone. Also find the limit.

P.T.O.

4. (a) Let X = (x_n) and Y = (y_n) be sequences of real numbers that converge to x and y respectively. Then the product sequence X.Y converges to x.y.

(b) Let X = (x_n) be a bounded sequence of real numbers and let $x \in R$ have the property that every convergent subsequence of X converges to x. Then the sequence X is convergent to x.

(c) Discuss the convergence of the sequence (x_n), where $x_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots\cdots + \frac{1}{2n}$ for $n \in N$.

(d) Use the definition of the limit of the sequence to find the following limits

(i) $\lim_{n\to\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right)$

(ii) $\lim_{n\to\infty}\left(\frac{3n+1}{2n+5}\right)$

5. (a) Prove that a necessary condition for the convergence of an infinite series Σa_n is $\lim_{n\to\infty} a_n = 0$. Is the condition sufficient? Justify with the help of an example.

(b) Prove that the geometric series $1 + r + r^2 + \cdots$ converges for $0 \leq r \leq 1$ and diverges for $r \geq 1$.

(c) Test for convergence, the following series :

(i) $\frac{1}{5}+\frac{2!}{5^2}+\frac{3!}{5^3}+\cdots$

(ii) $\sum_{n=1}^{\infty}\frac{\sqrt{n+1}-\sqrt{n-1}}{n}$

(d) Prove that the series $x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots$ converges if and only if $-1 \leq x \leq 1$.

6. (a) State and prove Cauchy's n^{th} root test for positive term series.

(b) Prove that the series $1+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}\cdots$ converges for $p > 1$ and diverges for $p \leq 1$.

(c) Test for convergence, the following series :

(i) $\sum_{n=1}^{\infty}\left[\sqrt[3]{n^3+1}-n\right]$

P.T.O.

(ii) $\sum_{n=1}^{\infty} 2^{-n-(-1)^n}$

(d) Prove that every absolutely convergent series convergent. Show that the series $\sum(-1)^n \frac{n+2}{2^n+5}x$ converges for all the real values of x.

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